# TEACHING LOGIC

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**ABSTRACT**: In this paper the author emphasizes the value of problems and problem solving in the teaching of logic. Afterwards, special attention is given to Natural Deduction. Also, this paper poses some reflections relative to the presentation of the notions of univocal connective and conservative extension in a teaching situation. Some brief remarks on Analytic Tableaux and logical terminology may also prove useful pedagogically.

**KEY-WORDS**: Teaching, Logic, Problem Solving, Natural Deduction, Analytic Tableaux.

## 1. INTRODUCTION

The author's goal in this paper is to share some of his experiences in the teaching of logic. It is hopefully written for teachers of logic interested in enriching their courses. Anyway, readers are warned that some of the possibilities referred to may only be implemented in the case of special audiences, e.g. students already conversant in mathematics. Readers are also warned that the author does not have much space here to give precise details of its contents. In section 2 problems are considered an interesting way to begin a logic course. It is easier for the students to grasp the concepts that logic introduces after they have themselves solved a problem that they usually find motivating and in whose solution the concepts that logic introduces are almost instinctively applied. In section 4 the author considers very briefly the matter of translation. Sections 5 and 6 are dedicated respectively to Natural Deduction and Analytic Tableaux. Some terminological points are taken up in section 7. In particular, the author does not consider the question of particular contents of different logic courses according to ages or especial interests of the students. This question is dealt with e.g. in [14]. Also the author will not say anything regarding computer-assisted logic teaching (for this the reader may see e.g. [8]).

## 2. PROBLEMS

To start with the author has found useful stating problems for the students to solve. Logic is about deduction, so the best way to begin seems to be to help students focus on deduction. To this end, instead of talking about deduction or trying to define it, it seems better to state a problem for the students to solve. The students should not feel frustrated if they do not solve it, but it is important that they consider it in earnest.

Problems should neither be trivial nor too difficult. Also, they should not have an easily seen mechanical solution. In some cases, they may be dealt with observing or manipulating concrete objects, such as coins or pieces of paper, something which can easily be done in a classroom. This provides for a connection with the real world.

The author uses the strategy of leaving the students alone with the chosen problem for (say) five minutes. Even if the students do not solve the problem almost whatever they do will be useful for their understanding of logic, because this is a situation where they can compare their own thinking with the theoretical material that logic introduces. This procedure can be repeated the first minutes of every class taking appropriately chosen problems whose solutions are relevant for the contents of the class.

Example of appropriate problems may be found in several books by several authors, e.g. Gardner in [5] and [6] and Smullyan in [13]. An example may be the very well known problem of finding out the color of the hat of a blind person belonging to a group of three persons, the other two with normal sight though no one can see his own hat, where every one has a hat taken from a set of five hats, two (say) red and three (say) white and all are consecutively asked about the color of their own hat. One of the persons with normal sight is asked first and he says he cannot deduce his own hat color after observing the hat of the others. The second person with normal sight is asked afterwards and he also says he cannot deduce his own hat color after seeing the hat of the blind and remembering what the previous person with normal sight has answered. Finally the blind deduces his own hat color having paid attention to the previous answers.

Solving problems has been emphasized also as the appropriate way of learning mathematics by authors such as Polya (see [9], [10] and [11]). And note that both problem solving and Polya are also mentioned in [14], but, strangely enough, only in the context of a strategy for ages between 10-13.

## 3. THE CONNECTION BETWEEN LOGIC AND THE SOLVING OF PROBLEMS

After having solved certain problem or after having done any work towards the solution of a problem, it will be possible to call the students attention to the fact that they have been using expressions such as “therefore”, “and”, “or”, “if, then”, “not”, etc. Here it seems important to make clear that what really matter are the concepts behind those words and not the words themselves. Logic takes its way with formal languages, but that should not be confused with the fact that the concepts or meaning of words are what really matter. It is also the opportunity to distinguish concepts of words such as “therefore” which will be dealt in the logical theory as a relation and concepts of words such as “and” which will be dealt as an operation.

## 4. TRANSLATING FROM A NATURAL LANGUAGE

Learning logic can partially be seen as learning a language. Teachers of a second language usually say that translating is not a good way of learning a new language. Nonetheless, teachers usually have a good experience in the teaching of logic making students translate between natural and logical language. But the students should be cautioned: *literal* translations do not work in many cases. So, the only available procedure in the case of translating into logical language seems to be to grasp the logical meaning of statements taking into consideration, for example, that propositional language is very poor in comparison with the natural languages.

Let us now consider a situation that is something more that just doing a translation and that seems to be very useful in the teaching of logic. The situation involves some manipulation of logical concepts, something more than a mere translation. Let us consider an example to illustrate this question. There are traditional problems in an island whose natives are either knights or knaves (see e.g. [13]). The knights always say the truth and the knaves always lie. In one such problem there are three natives (A, B and C) and a foreigner who asks A, "Are you a knight or a knave?". The answer of A is not understood. Afterwards B says "A said he is a knave". Then C says "Do not believe B, he is a liar". The question is, what are B and C? Now, instead of asking students to directly find the solution of the problem, one may ask them in a *first* moment, to describe the situation using natural language. In the example just given, the description may be given by considering true the following two biconditionals: 1) *B* is a knight if and only if (iff) A said he himself was a knave, and 2) C is a knight iff *B* is a knave. Now, the arriving to these two biconditionals is not a matter of translation. There has been some logical thinking to arrive to them, i.e. the students have considered what B and C have said in the light of the hypotheses that there two type of people in the island, etc. Moreover, 1) can be reworked to obtain 1') B is a knight iff (A is a knight iff A is a knave). The author believes that these workings are very important in the teaching of logic. *Afterwards*, in a *second* moment, students may proceed to obtain translations (or formalizations or symbolizations or whatever we call them). In our example, we get that the following two biconditional formulas should be taken as true: $q \leftrightarrow (p \leftrightarrow \neg p)$ and $r \leftrightarrow \neg q$, where p, q and r stand respectively for "A is a knight", "B is a knight" and "C is a knight". And, again, afterwards, i.e. in a *third* moment, the students may be asked to *calculate* the solution of the problem. In examples like the one just given it seems also relevant to call the students attention to the fact that in this way it is possible to formalize what the speakers say also when they say something about what the others have said.

## 5. TEACHING NATURAL DEDUCTION

Teaching Natural Deduction at an introductory course poses some difficulties. One of them is that the student has to choose between many rules. In order to simplify the situation, the author provides the following analogical situation he has found in [11]. There are three containers of, respectively, 8, 5 and 3 liters. The first is full of (say) water and the other two are empty. The goal is to divide the 8 existing liters in two parts (of 4 lts. each). But the containers do not have any level-indicators, so all one is allowed to do is to either fill or empty a container choosing the appropriate move each time until one reaches the goal. The analogy with a derivation is the following: the initial situation may be abstracted by the triple (8, 0, 0) and is like a formula given as hypothesis, the goal is the triple (4, 4, 0) that is like the formula that has to be derived and the moves of filling or emptying are like Natural Deduction rules. Students usually get engaged to this problem with fun. Then they are told that derivations are like sequences of triples, but instead of having to choose between only *two* moves, they will have to select the appropriate rule between many of them.

Anyway, the author has had a good experience distinguishing several stages in the teaching of Natural Deduction, both because it may be pedagogically convenient that the student does not have to deal simultaneously with all the rules, and also because it is an adequate situation to present several non-classical logics. The author distinguishes *seven* stages: conjunction, conjunction and disjunction, distributive, positive, minimal, intuitionistic and classical logic.

Conjunction logic is given by just the usual Introduction and Elimination rules for conjunction. Exercises are deriving associativity, commutativity and idempotence.

The conjunction and disjunction stage is obtained restricting the usual Elimination of Disjunction rule ($\varphi \vee \psi$, $[\varphi]\ldots\chi$, $[\psi]\ldots\chi$ / $\chi$) in the following way: in the two sub-derivations it is not allowed to use additional hypotheses other than the two sub-formulas of the disjunction being eliminated, e.g. in the sub-derivation $[\varphi]\ldots\chi$ one can only use $\varphi$ as hypothesis. This prevents distribution. In fact, people conversant with algebraic logic, may easily notice that the allowing for other hypotheses is equivalent to distribution.

Distributive logic is obtained strengthening the Elimination rule for disjunction, allowing for additional hypotheses in the sub-derivations, i.e. using the usual Elimination of disjunction rule.

After introducing the usual rules for the conditional we get Positive Logic and now derivable formulas without hypotheses are for the first time available.

An easy exercise at this stage is to derive the (usual) strengthened form of the Elimination of disjunction rule and distribution using the weaker form (used in the Conjunction and Disjunction Logic) and the rules for the conditional. This is an example of a non-conservative extension, that is, the conditional rules allow for deriving formulas that do not involve the conditional. A positive example of a conservative extension is adding disjunction to conjunction. This concept is the notion involved in the question sometimes posed by students whether derivations of exercises involving only certain connectives can be given just using the rules for those connectives, so at least the teacher should clearly understand what is behind those questions. A common situation is the perception that formulas like Peirce Law though having the conditional as only connective cannot be obtained just using the rules in conditional logic (i.e. the logic with language $\{\rightarrow\}$ and the usual Introduction and Elimination rules for the conditional), but also require e.g. the classical negation rules. This fact can be rigorously proven considering the valuation that comes from the Heyting Algebra $H_3$, that is, defining a three valued valuation for the conditional and observing that the conditional logic rules preserve the top (this is perhaps easier if the Introduction rule is axiomatized in the usual way) but Peirce Law gets a value distinct from the top.

Minimal Logic is obtained by adding the usual Introduction and Elimination rules for negation. It is to be noted that in this way negation is not really defined, as it is easy to see that there is some circularity between negation and the absurdo, which is not yet enough to derive every formula. This is perhaps the right moment to introduce the concept of univocal connective. We say that the *n*-ary connective *k* is *univocal* in the logic *L* iff $k\varphi_1\ldots\varphi_n \dashv\vdash k'\varphi_1\ldots\varphi_n$, where k´ has analogous rules to k. Easy exercises are seeing that conjunction, disjunction and the conditional are univocal in positive logic. Let us see this in the case of conjunction. The following derivation (together with the reciprocal analogous derivation) proves that conjunction is univocal:

1. $\varphi \wedge \psi$ Hyp
2. $\varphi$ E $\wedge$, 1
3. $\psi$ E $\wedge$, 1
4. $\varphi \wedge' \psi$ I $\wedge'$, 2, 3

In the case of minimal logic, it is also easy to see with appropriate valuations that negation is *not* univocal. Then it can also be easily seen that negation is univocal in intuitionistic logic, thus showing perhaps the crucial difference between minimal and intuitionistic logic.

When introducing intuitionistic logic it may be useful to note that in the context of minimal logic the usual EASQ rule is both equivalent to the traditional *Modus Tollendo Ponens* and to Robinson Resolution, i.e., $\varphi \vee \psi, \neg \varphi \vee \chi \,/\, \psi \vee \chi$. The latter may be relevant in the case of students interested in Computer Science.

In the case of classical logic it seems unavoidable to say something regarding the distinction between constructive and non-constructive proofs.

As an example of a paraconsistent logic that could be presented we mention P1. This logic can be seen as the fragment of classical logic with language $\{\rightarrow, \neg\}$, the usual Introduction and Elimination Rules for the conditional and the rules $[\neg\varphi] \ldots \neg\psi, [\neg\varphi] \ldots \neg\neg\psi \,/\, \varphi$ and $\varphi \rightarrow \psi \,/\, \neg\neg (\varphi \rightarrow \psi)$ for negation. Stating its rules may not be enough to motivate a logic, but we just mean to say that the Natural Deduction setting allows for an easy presentation of many non-classical logics.

Now, let us face the constructions of derivations. There is an interesting coin-puzzle that can be presented to the students because of some analogy in its solution with the construction of Natural Deduction derivations. It appears e.g. in chapter 2 in [6]. Let us place eight coins in a row (it will be more neat if we take coins of equal size). The goal is to arrange them in four piles of two coins each pile, but this has to be achieved in four movements, where each movement means jumping with one coin (in any direction) exactly over two coins (these coins may either be separated or already in the same pile) and placing the coin above the next coin (not yet doubled). Now, the interesting point is that if one tries to solve the puzzle “upwards-downwards”, i.e. trying to obtain the four piles *from* the starting point, one may last a long time. If, instead, the person thinks “downwards-upwards”, i.e. puts himself in the goal situation and asks himself “how could I arrive to this situation?” the problem becomes trivial! The analogy with derivations is that in many cases the process of constructing derivations is more easily dealt with if one proceeds “downwards-upwards”, i.e. asks himself “by what rule could I naturally arrive to certain formula?” The answer is usually given by selecting the introduction rule of the main connective of the goal-formula.

The concepts of univocal connective and conservative extension may perhaps be taught only at special occasions e.g. especially interested students that want to achieve a better understanding of the subject. They may be especially relevant if the teacher is interested in the teaching of non-classical logics. People interested in Philosophy are reminded that e.g. Belnap and Dummett consider the concepts of univocity and conservative extensions respectively in [1] and [3] (in the latter case see p. 217-220 and p. 246-247). In the paper [4] the concepts of univocal connective and conservative extension are presented in detail in a toy situation, i.e. when adding conjunction to a logic just consisting of propositional letters.

When teaching Natural Deduction it may also be convenient to mention the relationship with the algebraic approach to logic. The field of Algebra of Logic is very sophisticated and has been very fruitful in the XXth Century. One connection not difficult to state is between the notion of supremum and the concept of disjunction, because it is easy to define supremum in the context of a partially ordered set and this definition corresponds directly to the introduction and elimination rules for disjunction. Also it is easy to see in intuitionistic logic that absurdo works like a minimum (note by the way that the supremum is also the minimum of certain set). The relationship between infimum and conjunction is not so direct in the case of the usual Gentzen introduction rule for conjunction.

In Natural Deduction there is also the basic distinction between first-level and second-level rules of inference. The reader interested in this question may read [7].

## 6. ANALYTIC TABLEAUX

Natural Deduction in not appropriate for classical logic, because derivations become cumbersome, especially in the case of classical predicate logic. On the contrary, analytic tableaux are very convenient for classical logic. This method has the advantage of systematically looking for a derivation or a counter-example (it should be made clear that soundness and completeness taken together are equivalent to the existence of either a derivation or a counter-model). Of course, in the case of propositional logic one has decidability. In the case of the quantifier rules it is natural to expect finite terminating tableaux in the case of finitely-satisfiable sets of formulas, something that does not happen e.g. in the very popular case of the D-rules in [12] (see p. 53-54). Regarding this question the reader may profitably read [2], where the mentioned D-rules are slightly changed in order to get the desired finitely terminating Tableaux.

## 7. TERMINOLOGY

Nomenclature aspects may be important especially for first year students. These students are usually not prepared for usual natural language words used as technical terminology not having the usual natural language meaning. Also, in some cases, the words used as technical terminology are not the best choices. For example, the author thinks it is better to use the expressions “derivable formula” and “derivation” instead of “theorem” and “proof”, because he thinks that the latter are vestiges of logicism that may be confusing, reserving the latter expressions for the soundness theorem or alike mathematical facts and their proofs. As another example, it is better to abbreviate the intuitionistic rule with “EASQ” (where “A” stands for absurd) instead of EFSQ (where “F” stands for false).

Another question, partially terminological, is that the expression “counter- model” is very usual in predicate logic. Analogously, the expression “counter-valuation” may prove useful when teaching propositional logic, where the student should be trained to be able to either find a derivation or a counter-valuation. Also, as a general rule, it may be important to distinguish between “*counter*-examples” and “*non*-examples”, the former referring to objects that make false certain statement and the latter referring to objects that do not fall under certain concept. For example, according to the usual definition, the expression *(p* is a a *non*-example of a formula, that is, it is an expression that is not a formula. But 2 is a *counter*-example of the statement that every prime is odd.

A terminological point that has connections with philosophical aspects of logic is the use of the words "soundness" and "completeness" for the usual theorems in propositional and predicate logic. It seems to be a good idea to call the students attention to the historical fact that the previously mentioned words where chosen because the semantic approach to logic was given priority over the syntactical approach. Now, this priority is not a mathematical fact, e.g. there is no theorem asserting this priority. There are for instance some logicians that have warned that e.g. syntactical rules should not necessarily be considered meaningless. So, we have here a philosophical question some people may find interesting and over which much has been written.

We insert in this section a brief remark regarding non-monotonic reasoning. The author believes it is very important to mention the fact that usual daily reasoning is up to certain extent non-monotonic, whereas classical logic and the non-classical fragments mentioned in this paper are monotonic, including schemes such as $\psi \rightarrow (\varphi \rightarrow \psi)$ that seem puzzling to students. But, unfortunately, it seems impossible to include material on non-monotonic reasoning in the usual introductory courses to logic. It is probably a good idea to restrict oneself to the usual Tweety example.

## 8. WHAT ABOUT THE FUTURE?

How shall we be teaching logic in the future? Will we still teach classical logic? Will we be doing it in the same way we usually do it now? These and similar questions arise naturally. The author's guess which probably most readers will agree with is that classical logic will always have a predominant role in the basic teaching of logic. But the author's opinion regarding the teaching of non-classical logics, is that many of them that are fragments of classical logic are at least in part quite easily dealt with in the context of classical logic. The author has already given an idea of his procedure in the section of this paper dedicated to Natural Deduction and the concepts of univocity and conservative extension are some of the relevant notions involved in the classification of non-classical logics. Regarding the third question the author believes that perhaps the recent algebraic developments of logic will also have an important role in the future teaching of logic. But, as logicians seem to be endlessly refining their settings, a final solution is never to be expected.

## REFERENCES

[1] Belnap, N. Tonk, plonk and plink. *Analysis*, v. 22, 130-134, 1962.

[2] Boolos, G. Trees and Finite Satisfiability: Proof of a Conjecture of Burgess. *Notre Dame Journal of Formal Logic*, v. 25, n. 3, july 1984.

[3] Dummett, M. *The logical Basis of Metaphysics*, Harvard University Press, Cambridge, Mass., 1991.

[4] Ertola, R. and Galli, A. Algunos Conceptos para la Enseñanza de la Lógica. In: *Ciencias Formales y Filosofía* (ed: Andrés Bobenrieth), Edeval, Chile, 2007, p. 213-222.

[5] Gardner, M. *Mathematical Puzzles and Diversions*. Penguin, Hardmondsworth, Middlesex, England, 1965.

[6] Gardner, M. *Mathematical Carnival*. Alfred Knopf, New York, 1975.

[7] Makinson, D. Natural Deduction and Logically Pure Derivations. *ΦNEWS*, v. 5, 2004, p. 5-20.

[8] Manzano, María et alia (Eds.). *Proceedings of the Second International Congress on Tools for Teaching Logic*. Salamanca, Spain, 2006.

[9] Polya, G. *Mathematics and Plausible Reasoning*. Princeton University Press, Princeton, 1954.

[10] Polya, G. *Mathematical Discovery*. Wiley, New York, 1965.

[11] Polya, G. *How to solve it*. Princeton University Press, Princeton, 1957.

[12] Smullyan, R. *First Order Logic*, Dover, New York, 1995.

[13] Smullyan, R. *What is the Name of this Book?* Prentice-Hall, Englewood Cliffs, NJ, 1978.

[14] The ASL Commitee on Logic and Education. Guidelines for Logic Education. *The Bulletin of Symbolic Logic*, v. 1, n. 1, March 1995.