Critique of Kant on Arithmetic

Leslie STEVENSON[1]

In the Transcendental Aesthetic of the *Critique of Pure Reason* Kant claims two kinds of synthetic *a priori* knowledge – in mathematics, and in our representations of space and time as wholes.[2] He says that both kinds involve *a priori* (pure) intuitions; but they raise different issues, as he realized in his "prize essay" of 1764 when he first demarcated the methods of mathematics and metaphysics. After his "great light" in 1769 he distinguished intuitions, i.e., immediate acquaintance with particular objects affecting our senses, from concepts, i.e., general rules of classification (and constituents of judgments). The distinction between sensibility and understanding seems obvious now, but the notion of *a priori* intuition is a more obscure and controversial matter, as we will see.

Kant has some persuasive things to say about the primitive basis of arithmetic in our activities of counting and calculating. This is obviously where mathematics *starts*, when children learn their first concepts of number, and some people do not get much further. Many Kant-interpreters recycle his elementary examples of small addition sums (B15) and easy geometrical constructions (A716-7/B744-5).[3] But there was more to the mathematics of Kant's time, as he well knew,[4] and of course the subject has advanced by leaps and bounds since then. I want to discuss how well his account of arithmetic holds up in the light of later developments in philosophy, logic, and the foundations of mathematics. Geometry is a more natural home for Kant's distinctive claims, and many commentators mention arithmetic only briefly, and hurry on to discuss the intuitive evidence and necessity of Euclidean geometry with its obvious connection with space, and its supposed support for transcendental idealism.[5] This essay will deal with arithmetic only, in the hope that a careful examination of arithmetic as candidate for synthetic *a priori* status, and as topic of *a priori* intuition, may yield philosophical dividends.

https://doi.org/10.36311/2318-0501.2021.v9n1.p77

In the *Prolegomena* Kant remarked that whereas geometry is based on the pure intuition of *space*, arithmetic forms its concepts of numbers by the successive addition of units in *time* (4:283), and that suggests a neat division of duties. But though we can count a succession of temporal episodes, such as the tolling of a bell or the bars in a piece of music, we can also recognize at a glance small numbers of static objects such as people in a room or marks on a page; moreover, we can count "things" that are not in space or time, for example the four prime numbers between 10 and 20. Counting is a mental procedure that calls on knowledge of an ordered set of numerals (at least, the first few of them). Kant emphasized that any process of counting takes time (perhaps a long time: if imprisoned, one might record the days by making scratches on the wall).

Once one can do some counting, elementary addition becomes possible. But in the B-Introduction Kant controversially declared as "incontrovertibly certain" and "very important in the sequel" that all mathematical judgments are synthetic.[6] He admitted that mathematical *inferences* "proceed in accordance with the principle of contradiction" (B14) and that "equals added to or subtracted from equals are analytic propositions" (A164/B204), yet he stoutly maintained that the *propositions* of pure mathematics are *synthetic*, even simple sums such as 7+5=12:

> The concept of sum of 7 and 5 contains nothing more than the unification of both numbers in a single one … The concept of twelve is by no means already thought merely by thinking of that unification of both numbers in a single one, and no matter how long I analyze my concept of such a possible sum I will not find twelve in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or … five points, and one after another add the units of the five given in the intuition to the concept of seven … The arithmetical proposition is therefore always synthetic … (B15, see also A164/B295 and *Prolegomena* 4:268-9)

A crude reaction to this is that for numbers greater than ten you will run out of fingers, and if you have lost a finger or two in an accident your counting will run into problems even earlier. And if you resort to making marks on paper you had better beware of slips of the pen, or ink running out: any such reliance on *empirical* intuition, i.e. perception of the physical world, is liable to all the shocks that such stuff is heir to. In olden days people could use an abacus, nowadays we have electronic calculators; but all material apparatus is subject to user error or malfunction.

A classic worry arises from the vagueness of Kant's notion of analyticity, appealing to what is "contained" or "thought" in a concept. But neither of those criteria can decide what is analytic or synthetic in controversial cases. At B17 he says "the question is not what we *should think* in addition to the given concept, but what we *actually think* in it, though only obscurely" – but that only makes the matter more obscure. Kant offered something clearer later on, in his introduction to the Analytic of Principles:

> If the judgment is analytic, whether it be negative or affirmative, its truth must always be able to be recognized sufficiently in accordance with the principle of contradiction. (A151/B190, see also *Prolegomena* 4:267)

The test is now whether the negation of the judgment implies a contradiction. That will still not decide the question when there is no consensus about what is "contained in" or implied by ordinary, non-mathematical concepts, such as gold or marriage. However, there should be better prospect of applying the contradiction test in mathematics, where we deal with strict definitions. As Kant himself said:

> Mathematical definitions cannot err. For since the concept is first given through the definition, it contains just that which the definition would think through it. (A731/B759)

Of course, one can think of, and write in symbols, an addition sum of two large numbers, such as 342597 + 68069, without there and then *thinking* of the answer, the number which is their sum.[7] But one may be equally foxed by a complex chain of syllogisms or a long formula in propositional logic, even if the inference is valid or a truth-table will show that the formula is a tautology: and those surely are logical truths, and presumably therefore *analytic*.[8] Not all logical truths are obvious, and those that are not can be decided by various technical procedures, which Kant would call "constructions". But is there any tension between analyticity and reliance on such constructions?

To apply this to Kant's example, here is a backwards method to show that 7+5=12 is analytic by the contradiction test. Suppose that 7+5 is *not* equal to 12, then subtract 1 from 12, and 1 from 5, then from 11 and 4, and so on, and you will quickly reach the contradiction that 7 Is not equal to 7.[9] You don't need decimal notation to do this, and you don't have to rely on your fingers, you only need to count up to twelve, then go back five steps, one at a time. Our words for numbers are learnt in whatever language our parents speak: we repeat the first ten or twenty numerals by rote in *the right order*, then we use them to count small sets of objects or events; later we learn the written numerals up to 9, then we catch on to the use of zero in decimal notation, and may come to realize that it provides for indefinite extendibility[10]. One of our earliest lessons is that the word 'five' comes after 'four' in the series. That is an empirical fact about English usage, but 'four plus one is five' and '5-1=4' surely qualify as true by definition. *All* uses of words or symbols rest on conventions, of course, but that does not mean that *every* sentence or formula is synthetic. Kant recognized that in mathematics we use precise definitions about how to use our mathematical words and symbols (see the opening of the 1764 prize essay at 2:276, and A726-732/754-760). But that does not make arithmetical judgments empirical; we assert them *a priori,* only on the basis of having learnt the relevant conventions, but *that* applies whenever we say anything about anything.[11]

It is tempting to conclude that the ordered list of numerals plus the rule-governed procedures for addition, subtraction and multiplication, make arithmetical truths analytic, in the sense that their negations imply contradictions. However, subtle questions lurk about the nature of logical implication and logical truth in mathematics. Kant's understanding of logic was primitive by our post-Fregean standards, and that was arguably his greatest intellectual handicap. He knew only the theory of syllogisms supplemented by disjunction and conditionality; he had no notion of the predicate calculus with its relational terms and quantifiers. So, it is understandable why he thought that even simple arithmetical sums called on extra-logical resources, and had to be classed as synthetic.

Many logicians and philosophers have since argued that our powerful modern logic shows that arithmetic is (non-trivially) analytic after all (and hence, perhaps, much more of mathematics). But Kantian-inspired doubts have been raised about this by Parsons (1983) and Friedman (1992).[12] Frege and Russell, and others down to the present day, have pursued the logicist programme of defining numbers in terms of logic, and deriving all arithmetical truths from logical truths.[13] But despite a century of immense technical sophistication,[14] nagging questions remain about what exactly qualifies as logic, and hence as analytic. Does logic include set theory, or higher-order quantification, and can the existence of an infinity of numbers be derived from logic alone? Although mathematical proof aspires to complete logical rigour, and surely attains it in arithmetic, there is still room for Kantians to maintain that the underlying assumptions that make arithmetic possible are not *purely* logical truths even by modern standards.

Once we can count sets of manageable size, by pronouncing or marking numerals while we correlate them with distinguishable objects of attention we are counting,[15] we take a new step when we understand that there is no end to the series of numbers, that there always *is* (in a remarkably abstract sense of 'existence') a number greater than any number we have so far thought of or symbolized. Our symbols can get ever longer, but they remain necessarily finite; yet according to our conception of the integers, every number has a successor, with no repeats, so there are infinitely many of them. Whether this is a *logical* truth depends, as noted already, on what we are prepared to describe as logic. Even that little reverse proof that 7+5 = 12 has to proceed step by step: 12-5, 11-4, 10-3, 9-2, 8-1, 7; and to be recognized as valid, the procedure has to be carried out *in time* (like the literal steps in walking). Kant had some reason, then, to say that though arithmetical truths are not *about* anything happening in time (their *content* is not temporal) our ability to recognize them depends on our ability to complete iterative mental processes that take time.

Does this vindicate Kant's notion of *a priori* intuition? In his reply in 1790 to Eberhard's Leibnizian criticisms he offered a more positive account of syntheticity:

> the principle of synthetic judgments in general, which follows necessarily from their definition, [is] *that they are only possible under the condition that an intuition underlies the concept of their subject*, which, if the judgments are empirical, is empirical, and if they are synthetic judgments *a priori*, is a pure intuition *a priori*. (1790, 8:241, with Kant's emphasis)

Allison has commented that it is the notion of synthetic that "wears the trousers" here, since analytic judgments could now be characterized as those whose truth does *not* depend on intuition, so that any reference to an "object" is otiose, whereas synthetic judgments extend our knowledge in a "material" sense, asserting the "reality" of the predicate as applying to the subject (Allison (2004), pp.89-93). But it is not clear how such talk applies in arithmetic: it seems to lead into unanswerable questions of mathematical ontology, perhaps merely verbal disputes. Are numbers objects? Do we refer to them? Are arithmetical predicates real? Can arithmetic extend our knowledge materially? I hardly know how to address these questions, but I do want to ask whether arithmetic involves *a priori* intuition.

Kant's late gloss on syntheticity, as quoted above, appeals to his bifurcated notion of intuition. If we were to ignore his invocation of *a priori* intuition, we would have only the usual distinction between analytic truths and synthetic facts knowable by perception, as in Hume and logical positivism. Kant doggedly maintained his third realm of the synthetic *a priori* as the basis of his critical philosophy; but the above account threatens to tie our understanding of synthetic *judgment* to the notion of *a priori intuition*. Is he making arithmetic analytic by the contradiction test of 1781, but synthetic by its dependence on *a priori* intuition in 1790? That would be very confusing, we had better take him at his word in the latter passage that he is not there *defining* syntheticity, only pointing out what his philosophical views commit him to saying about it. That puts the pressure on his notion of *a priori* intuition: but that is a paradoxical notion, for there is a *prima facie* tension between being a quasi-perceptual experience and being prior to experience. My discussion here is restricted to its relevance to arithmetic.

It is clear that doing arithmetic makes mental demands on us, at various levels. Animals and infants cannot count. Children learn the first few number words, and use them to count small sets of objects; later they begin to apply the rules for addition, subtraction and multiplication (division can present more of a challenge). At any stage some will encounter "learning difficulties", and many adults remain mathematically challenged.[16] Many of us can do a little computation "in our heads", but at some point we need to do our sums on paper. A few of us study mathematics, and a select few become mathematicians.

An important part of Kant's point is that in mathematics we find it much more efficient, and in a sense "intuitive", to use visible symbols rather than words.[17] Arithmetical propositions can in principle be expressed in words, if one is prepared to use enough of them, but it is much more convenient to use systems of symbols. The Romans had an unwieldy notation in which 4 was represented, not by 'IIII', but by 'IV'. The Arabs had the brilliant idea of a symbol for zero, which made possible the decimal system that is indefinitely extendible.

To illustrate the utility of symbolic notation in mathematics, I have set out in an appendix the elegant little proof that there is no rational square root of two, firstly in algebraic symbols, then in words, then in logical notation. The propositions expressed do not depend on the language used (English or Japanese) or the choice of symbols (decimal or binary). The verbal and logical versions can be formulated for this short argument, at least, but the symbols such as $(a/b)^2 = 2$ make it much easier to follow. The words and logic are cumbersome and "unintuitive" in their own way, even in this easy case, so it is obvious why most mathematical reasoning is conducted in whatever symbolism is found convenient for the job in hand. Heavy-duty logical apparatus can in principle be wheeled in to *check* whether a putative proof is completely watertight, testing it line by line for conformity to recognized rules of inference.[18] But it is a crucial point that there are no mechanically-applicable rules for *constructing* proofs, and that recently-vindicated thought provides another partial vindication of Kant's notion of *a priori* intuition.

Arithmetic may be narrowly defined as "sums," i.e. addition, subtraction multiplication and division, involving only the integers. These procedures *can* be mechanized, for the relevant questions are computable in the technical logical sense (though computers have physical limits whereas numbers do not). Many new mathematical concepts have been developed: we now recognize negative integers, rational numbers, real numbers, and complex numbers (counting on your fingers won't take you very far with these). It is a merely verbal question whether all that can be called "arithmetic," but Kant knew that there is more to number theory than primary school sums. We can define further arithmetical concepts such as prime number, perfect number, and square root, and in terms of them we can formulate some interesting general propositions. Some can be readily proved, e.g., that there is no greatest prime number; others remain unsolved, e.g., the Goldbach conjecture that every even number is the sum of two primes; and some, though easy to understand, have been proved only by appeal to recently-developed esoteric regions of mathematics, e.g., Fermat's Last Theorem.

Kant declared at A713/B741 that mathematics uses "construction of concepts" rather than "rational cognition from concepts" as in philosophy,[19] and that seems to be his main reason for holding that mathematical judgments are synthetic *a priori*. But our breif survey of mathematical practice shows that there is more than one kind of construction:

1. There is the elementary stage of counting small sets of objects or events.
2. There is the stage of doing sums according to the standard rules, using symbols on paper (or these days, electronic calculators). These procedures are mechanical, and are guaranteed to produce the right answer if applied correctly.
3. Realizing that the numbers go on forever, beyond all physically constructible notations, and hence the validity of proof by mathematical induction.
4. Proving general conjectures about numbers. But because there are no mechanical methods for constructing such proofs; a higher degree of insight is required to understand the validity of the inferential steps, let alone to construct a proof one hasn't seen before.
5. More advanced mathematics uses a panoply of special symbols for complicated new concepts, and employs some esoteric methods of proof which only professionals working in the relevant area may understand. Mathematicians form new concepts, and at the highest degree of creativity they construct new kinds of proof. A mathematician can work for years to construct a proof before realizing, sometimes in a flash of "intuition", that a certain method will work.[20]

At A717/B745 Kant distinguished "symbolic" construction in arithmetic from "ostensive" construction in geometry;[21] and I am pointing out that there are *several more* kinds of construction within arithmetic and number theory.

Do these all involve "*a priori* (or pure) intuition?" The phrase is Kant's, but as far as I know he did not distinguish these different levels of its application.[22] Since Frege's refutation

(1884/1950) of Mill's implausibly radical empiricism about arithmetic, everyone agrees that the arithmetical propositions are *a priori*, i.e., no possible experience can count against them. A question arises, however, about Kant's extension of his notion of "intuition" from sense-perception of the material world to the recognition of mathematical truths. He can find some ordinary-language support from the fact that we can talk of "seeing" that 7+5 = 12, and at a more demanding level, of a mathematician "perceiving" that a certain method of proof will succeed. But what is *in common* between material and mathematical perception? In both there is recognition of objective truths that hold independently of anyone's mental states; and such judgments are not purely passive, not reflex reactions or mere associations of ideas, they involve what Kant calls "spontaneity" in applying concepts, whether empirical or mathematical. And judgments are often the result of activity and effort, to scrutinize a material state of affairs more closely, or to perform a computation or construct a proof. If mathematics can be said to involve some sort of *intuition*, it certainly involves concepts.

But there remains an obvious difference between sensory and mathematical perception, for the former involves, by definition, the causal impact of the external world on our sense-organs. To be sure, in doing arithmetic we may *see* symbols on the page or blackboard, we *hear* what our teachers say, and the blind can read braille by *touch*; but mathematical perception is more than seeing marks (as a chimpanzee might), it means "seeing *that*" certain necessary connections hold, judging the relevant propositions to be true *a priori*; and such *seeing that* can occur without any sensory perception at the time, in moments of mathematical insight, whether elementary or virtuosic. Kant acknowledged the causal element in intuition when he wrote:

> [intuition] takes place only insofar as the object is given to us; but this in turn [*at least for us humans* – phrase added in B] is possible only if it affects the mind in a certain way. … Objects are therefore given to us by means of sensibility, and it alone affords us intuitions. (A19/B33)
>
> It comes along with our nature that intuition can never be other than sensible, i.e., that it contains only the way in which we are affected by objects. (A51/B75)

Taken out of context, these sentences restrict "intuition" (*Anschauung)* to sense-perception, and would rule out its application to mathematical perception. But Kant goes on to talk of pure or *a priori* intuition which does *not* involve sensation, in which "the pure form of sensible intuitions in general is to be encountered in the mind *a priori*" (A20/B34). Yet at the beginning of the chapter on Phenomena and Noumena he wrote:

> Now the object cannot be given to a concept otherwise than in intuition, and even if a pure intuition is possible *a priori* prior to the object, then even this can acquire its object, thus its objective validity, only through *empirical* intuition [my emphasis], of which it is the mere form. … the concept of magnitude seeks its standing and sense in number, but this seeks this in turn in the fingers, in the beads of an abacus, or in strokes or points that are placed before our eyes. (A239/240/B298-9)

Kant is obviously right that this is where arithmetic *starts*, in the practice of counting, which involves *empirical* intuition plus a kind of conceptualization that is not available to animals (so far as we know). But it is equally obvious that the theory of numbers does not *end* there. Some faithful Kantians do not seem to take the point: Lucy Allais, for example, writes:

> It is crucial to Kant's account that a construction in intuition is not a theoretical (conceptual) construction: it essentially involves an object that is presented (exhibited, manifestly displayed). … Kant thinks that mathematical truth does not go beyond what can possibly be constructed in intuition, and this means that it is limited to something that can be manifested to consciousness ... Allais (2015, p. 203)

But what about the mathematical truth that there are infinitely many whole numbers – or the *greater* infinity of the real numbers, or the infinity of different infinities that have since been conceptualized? These are surely "theoretical (conceptual) constructions", but where is the "object" that is presented or "constructed *in intuition*"? To be sure, mathematicians use perceptible symbols to express these truths, but it is impossible to "construct" or "exhibit" an infinite set of symbols. We *believe* that for any given number there are infinitely many more, and we can "*see*" why that has to be true. But Kant's insistence on "pure intuition" in mathematics does not seem to add anything to the uncontroversial observation that we make certain conceptual definitions and constructions, and recognize the necessity of mathematical inferences and truths.

Kant asserted that the existence of synthetic *a priori* truths has momentous philosophical implications. In his "Transcendental exposition of the concept of space" (B40-1) he appealed to the (alleged) synthetic a priority of geometry to argue that geometrical intuition "has its seat merely in the subject". But in the supposedly parallel "Transcendental exposition of the concept of time" (B48-9) he does not explicitly appeal to *a priori* intuition in arithmetic, nor does he argue there that time is merely in the subject. In the First Introduction to the *Critique of Judgment* he admitted that "the universal theory of time, unlike the pure theory of space (geometry) does not provide us with enough material for a whole science" (10:237). The fact that we begin arithmetic by counting is hardly enough to prove the remarkable metaphysical claim that time "is nothing except the form of our inner intuition" (A37/B54), which can exist only "in us" (A42/B60).

I have not considered the cases of geometry or metaphysics in this essay, but as far as arithmetic is concerned, I have argued that the Kantian assertion that it is synthetic now rests on uncertainty about what we should count as logic. Unclarity about that suggests that we should not rest much philosophical weight on the claim that arithmetic is synthetic. I have also argued that Kant's conception of *a priori* intuition in arithmetic is more controversial than the notion of mathematical perception. Whatever the virtues (or vices) of *a priori* intuition in the rest of his philosophy, I do not think it casts much light on our concepts and practices in dealing with numbers.

APPENDIX: Proving that there is no rational square root of two

The argument in symbols

Suppose there were numbers a and b, with no common divisor, such that $(a/b)^2 = 2$

If so, $a^2 = 2b^2$, hence 2 must divide a, so for some number c, a= 2c

Hence $4c^2 = 2b^2$, so $2c^2 = b^2$, hence 2 must divide b

Then a and b would have a common divisor, namely 2

But that contradicts the original supposition, which is therefore false.

The argument in words

Suppose there were two numbers with no common divisor, such that the square of the ratio of the first number to the second number is two

If so, the square of the first number is equal to twice the square of the second number,

Hence two divides the first number, so it must be twice some third number

Hence four times the square of that third number equals the square of the second number

Hence two divides the second number

Then the first and second numbers would have a common divisor, namely two

That contradicts the original supposition, which is therefore false.

The argument formalized in logic

$(a/b)^2 = 2$ & --(Ex)(Ey)(Ez)(a = xy & b = xz)	hypothesis
$a^2 = 2b^2$	from 1
(Ex)(a = 2x)	from 2
$4c^2 = 2b^2$	from 2 and 3
(Ex)(b = 2c)	from 4
(Ex)(Ey)(Ez)(a = xy & b = xz)	from 3 and 5
(Ex)(Ey)(Ez)(a = xy & b = xz) & - (Ex)(Ey)(Ez)(a = xy & b = xz)	from 1 and 6
-- [$(a/b)^2 = 2$ & --(Ex)(Ey)(Ez)(a = xy & b = xz)]	*reductio ad absurdum*, 1 and 7

Abstract: Arithmetical truths are *a priori*, but our understanding of them starts with the practical experience of counting. Whether arithmetic is analytic – or synthetic as Kant maintained – turns out to depend on what view we take about the precise scope of logic. A survey of mathematical theorizing about various kinds of numbers shows that there is more than one kind of "construction" or "intuition" involved. Kant's conception of *a priori* intuition, as applied to arithmetic, seems to be just mathematical perception.

Keywords: *A priori*, Analytic, Arithmetic, Construction, Intuition, Logic, Mathematics, Perception, Proof, Synthetic

References

Allais, Lucy (2015) *Manifest Reality: Kant's Idealism and his Realism*. Oxford: Oxford University Press.

Allison, Henry (2004) *Kant's Transcendental Idealism: An Interpretation and Defense* (revised and enlarged edition). New Haven: Yale University Press.

Bennett, Jonathan (1966) *Kant's Analytic*. Cambridge: Cambridge University Press.

Carey, Susan (2011) *The Origin of Concepts*. Oxford: Oxford University Press.

Frege, Gottlob (1884/1950) *The Foundations of Arithmetic*, translated by J. L. Austin. Oxford: Basil Blackwell.

Friedman, Michael (1992) *Kant and the Exact Sciences.* Cambridge, Mass and London: Harvard University Press.

Hanna, Robert (2015) *Cognition, Content, and the A Priori: A Study in the Philosophy of Mind and Knowledge*. Oxford: Oxford University Press.

Kant, Immanuel (1763) *The Only Possible Argument in Support of a Demonstration of the Existence of God,* in *Theoretical Philosophy 1775-1770,* translated by David Walford and Ralf Meerbote. Cambridge: Cambridge University Press 1992.

Kant, Immanuel (1764) *Inquiry concerning the Distinctness of the Principles of Natural Theology and Morality,* in *Theoretical Philosophy 1775-1770*, translated by David Walford and Ralf Meerbote. Cambridge: Cambridge University Press 1992.

Kant, Immanuel (1781/1787) *Critique of Pure Reason*, translated by Paul Guyer and Allen Wood. Cambridge: Cambridge University Press 1998.

Kant, Immanuel (1783) *Prolegomena to any Future Metaphysics that will be able to come forward as Science*, in *Theoretical Philosophy after 1781*, translated by Gary Hatfield and Michael Friedman. Cambridge: Cambridge University Press 2002.

Kant, Immanuel (1790) *Critique of the Power of Judgment*, translated by Paul Guyer and Eric Matthews. Cambridge: Cambridge University Press 2002.

Kant, Immanuel (1790) *On a discovery whereby any new critique of pure reason is to be made superfluous by an older one*, in *Theoretical Philosophy after 1781*, translated by Gary Hatfield and Michael Friedman. Cambridge: Cambridge University Press 2002.

Kant, Immanuel (1992) *Lectures on Logic*, translated and edited by Michael J. Young. Cambridge: Cambridge University Press 1998.

Parsons, Charles (1983) "Kant's Philosophy of Arithmetic" and "Postscript," in *Mathematics in Philosophy: Selected Essays.* Ithaca: Cornell University Press.

Potter, Michael (2000) *Reason's Nearest Kin: Philosophies of Arithmetic from Kant to Carnap*. Oxford: Oxford University Press.

Quine, Willard (1953) "Two Dogmas of Empiricism", in *From a Logical Point of View*. Cambridge, MA: Harvard University Press.

Shabel, Isa (2006), "Kant's Philosophy of Mathematics," in Paul Guyer (ed), *The Cambridge Companion to Kant and Modern Philosophy*, Cambridge: Cambridge University Press, pp. 94-128.

Sutherland, Daniel (2017) "Kant's Conception of Number", *Philosophical Review* 126 (2), pp.147-190.

Tennant, Neil (2017) "Logicism and Neologicism," *The Stanford Encyclopedia of Philosophy* (Winter 2017 Edition), Edward N. Zalta (ed.),

URL https://plato.stanford.edu/archives/win2017/entries/logicism/

Notes

1 Lecturer, then Reader, in Logic & Metaphysics at the University of St.Andrews, Scotland 1968-2000 (now retired). He is author of *Inspirations from Kant* (2011), *Eighteen Takes on God* (2000), and co-author of *Thirteen Theories of Human Nature* (2017), all published by Oxford University Press, New York.

2 The Analytic of Principles claims a *third* kind of synthetic *a priori* knowledge arising from the Transcendental Deduction of the Categories, which I will not consider here.

3 A recent faithful interpreter is Shabel (2006).

4 In 1763 Kant remarked on the "numberless beauties" of Euclidean geometry, citing some non-trivial theorems (2:93-5) – but contrary to his own emphasis on constructions, he there chose to express those proofs in cumbersome words rather than in diagrams. At A169-170/B211-2 his mention of "flowing magnitudes" (*fliessende Grössen*) suggests some acquaintance with Newton's fluxional calculus, see Friedman (1992) pp.74-7. But how well he understood its details (or Leibniz's version) is another matter.

5 For example, Allais (2015) provides an impressive in-depth discussion of central themes in Kant's theoretical philosophy, but arithmetic and time do not appear in the index – perhaps because they cannot bear the greater weight that Kant puts on geometry and space.

6 As he already asserted in in his prize essay of 1764.

7 There are people (once called "idiots savants") who can do such calculations in their head instantaneously, but even they must have their limits when numbers run into trillions.

8 Bennett said that putative synthetic *a priori* judgments can turn out to concern "certain very complex and unobvious conceptual facts" (1966, p.14). Potter has made a similar point in more detail (2000, p.39).

9 From what Kant said at A164/B204 it surely follows that subtraction of equals from both sides of an *in*equality must preserve inequality.

10 Unlike that cumbersome Roman numerals, which soon run out of letters.

11 For a robust in-depth defence of the analytic-synthetic distinction against Quine's classic attack see Hanna (2015, Chapter 4). He also defends a Kantian notion of the synthetic a priori.

12 I can hardly emulate the depth and length of these modern classics, but I hope to cast a little more light from a different angle.

13 See Potter (2000), Chapters 2, 4 and 5 for a critical account.

14 For a very detailed review see Tennant (2017).

15 Sutherland (2017) has presented a very comprehensive analysis of the cardinal and ordinal aspects of Kant's conception of number.

16 The psychologist Susan Carey presents evidence that a primitive sensitivity to number is part of what she calls "core cognition," innately emerging in early infancy (2011, Chapter 4). She goes on to argue that the notions of the natural numbers and of rational numbers are human constructions that go beyond core cognition, by a process of conceptual development that (following Quine) she calls "bootstrapping" (2011, Chapters 8 and 9).

17 Presumably the blind can do some arithmetic, but can Braille enable a blind person to understand a mathematical proof by feeling the layout of symbols on a page, having them "exhibited" to touch instead of sight? There are interesting empirical questions here.

18 I doubt that most referees for papers submitted to mathematical journals actually resort to symbolic logic in their evaluations. It would probably be too cumbersome, and hence liable to at least as much error in application as informal "intuitive" logic. And there can be controversy over certain patterns of inference: for example, the rejection by mathematical intuitionists (in another sense of that controversial word!) of certain forms of proof in classical mathematics.

19 See also the Vienna Logic at 797-8 and 893, the Dohna-Wundlachen Logic at 697, and the Jasche Logic at 23, all in Kant (1992).

20 A famous recent example is Andrew Wiles's experience, dated as precisely as the morning of 19 September 1994, when he suddenly realized how to overcome the flaw in his first attempt at proving Fermat's last theorem. Of course, the flash of insight or intuition has to be verified by the detailed work of constructing and writing the proof in a public form that other mathematicians can evaluate.

21 See Friedman (1992), p.108.

22 Parsons (1983) p.141, distinguished the intuition involved in recognizing that 2+2=4 from that involved in proving by mathematical induction that something holds for all numbers.

Received /Recebido: 9.4.2021

Approved / Aprovado: 30.4.2021